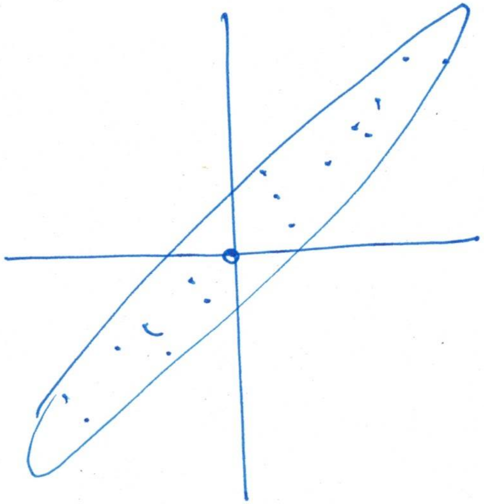
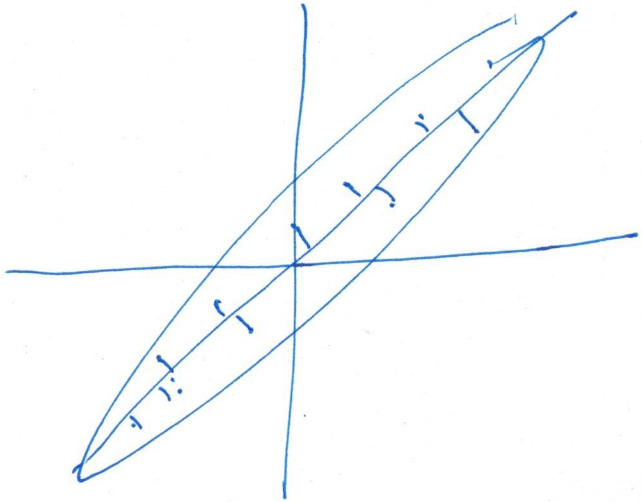
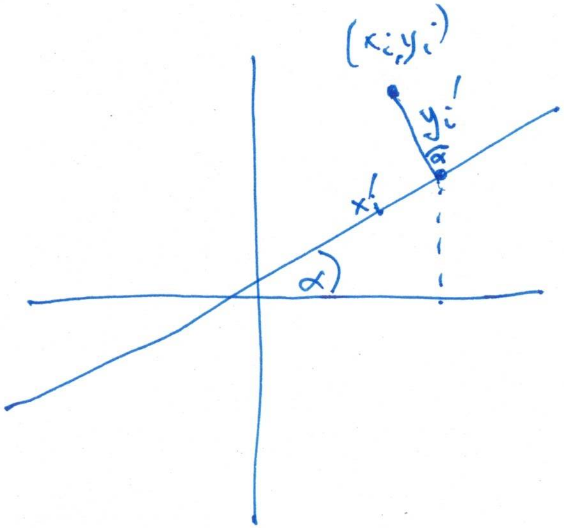






$$x_i = x_i' \cos\alpha - y_i' \sin\alpha$$

$$y_i = x_i' \sin\alpha + y_i' \cos\alpha$$

What is the rotation
that minimizes $\langle y^2 \rangle$

Maximize $\langle x'^2 \rangle - \langle y'^2 \rangle$!

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = R \begin{pmatrix} x_i' \\ y' \end{pmatrix} \qquad \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_i' \\ y_i' \end{pmatrix}$$

$\langle x_i' y_i' \rangle = 0$ uncorrelate x, y.

$$\begin{pmatrix} x_i' \\ y_i' \end{pmatrix} = R^{-1} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \qquad \begin{pmatrix} x_i' \\ y_i' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

$$\langle x_i' y_i' \rangle = \langle (x_i \cos\alpha + y_i \sin\alpha)(-x_i \sin\alpha + y_i \cos\alpha) \rangle$$

$$= 2 \sin\alpha \cos\alpha \left[\langle x^2 \rangle - \langle y^2 \rangle \right] + 2 \langle xy \rangle \cdot \cos 2\alpha = 0$$

$$\operatorname{tg} 2\alpha = -2 \cdot \frac{\langle xy \rangle}{\langle x^2 \rangle - \langle y^2 \rangle}$$

Maximizes

$\langle x^2 \rangle - \langle y^2 \rangle$

Inertial moment in physics

Becomes diagonal

L2

X : data matrix $X \in R^{n\times m}$

n features

m rows

$$X = \underset{\underset{\text{ortho normal}}{\nearrow}}{U}\,\Sigma\,\underset{\underset{\text{ortho normal}}{\nwarrow}}{V}^T$$

SVD

Singular Value Decomposition

$U^T U = \mathbb{1}$ $V^T V = V V^T = E$

$U \in R^{n\times n}$

$V \in R^{m\times m}$

$$\begin{pmatrix} \text{—} \\ \text{—} \\ \text{—} \\ \text{—} \end{pmatrix}\begin{pmatrix} | & | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} = E$$

$\Sigma \in R^{n\times m}$, diagonal

$$\Sigma = \left(\begin{array}{cccc} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \\ \hline & 0 & & \\ & 0 & & \\ & 0 & & \end{array}\right)$$

Rank

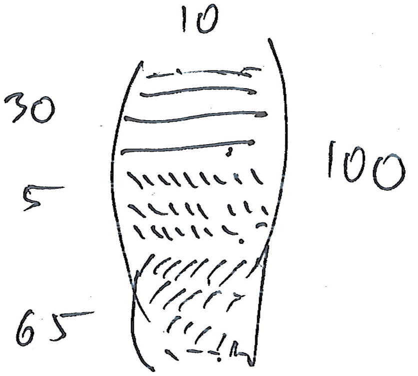
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -3 & 0 & 1 \\ 2 & 1 & 1 & 5 \\ -1 & -2 & -3 & -4 \end{pmatrix}$$

SVD

$$X = U\Sigma V^T$$

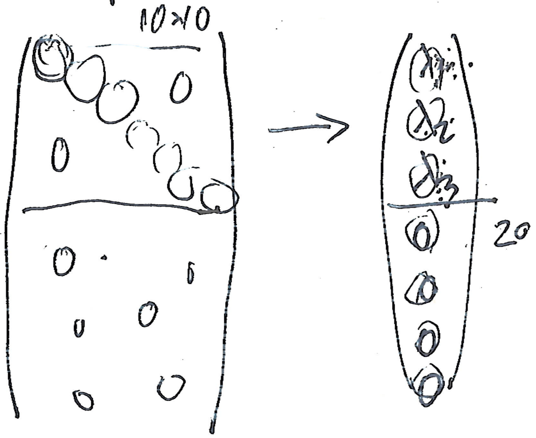
Rank = 3

$$N^2 \cdot N = N^3$$

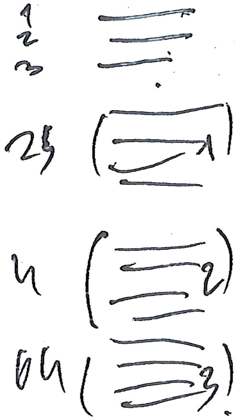


10^3

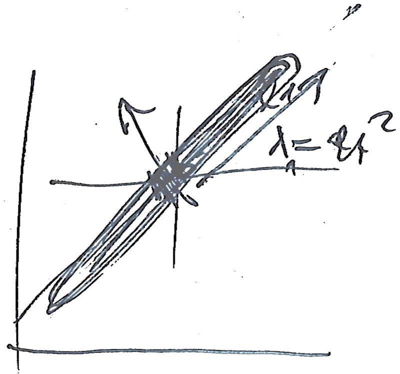
100^3



Rank = 3



$$\Sigma' = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \end{pmatrix}$$



$$S = \begin{pmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{pmatrix}$$

$$C_x = X^T X \quad (10 \times 10) \quad C'_x = X X^T$$

$$C_x = X^T X = (U\Sigma V^T)^T (U\Sigma V^T)$$

$$= (\cancel{V^T}\Sigma^T \cancel{U^T})(U\Sigma V^T) = (V\Sigma^T U^T)(U\Sigma V^T)$$

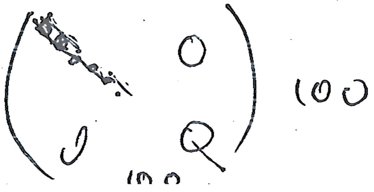
$$= V\Sigma^T \underbrace{(U^T U)}_{I} \Sigma V^T = V \underbrace{(\Sigma^T \Sigma)}_{S} V^T$$

$$\boxed{C_x = \cancel{U} V S V^T}$$

positive definite

$$\boxed{C_x = U^T S' U}$$

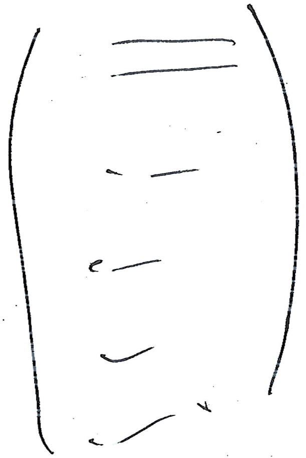
$$S' = \Sigma\Sigma^T$$

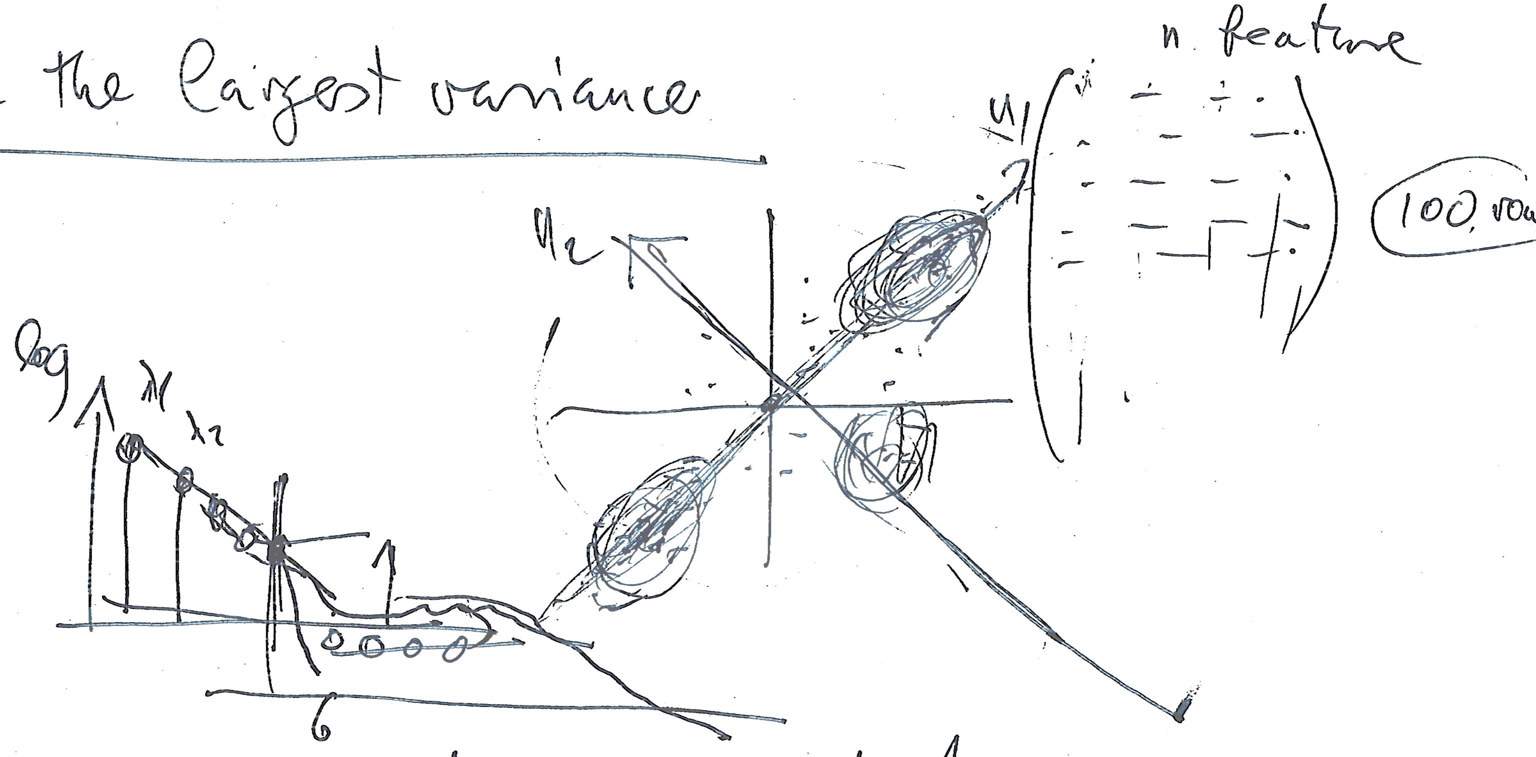


Directions of the largest variance

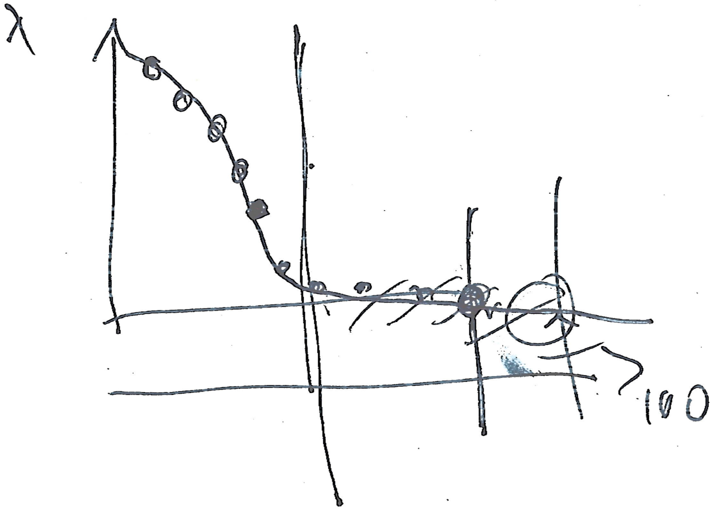
Face 1000 pixels

100,000 faces



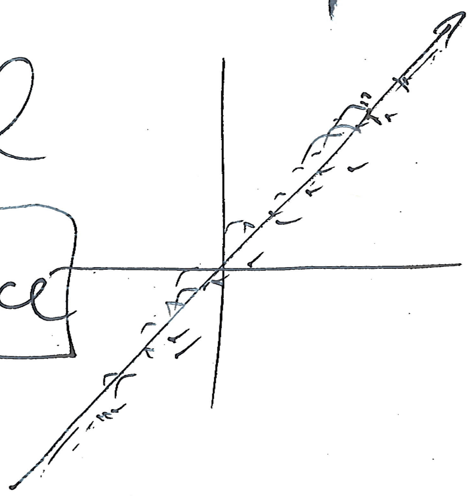


Truncated representations

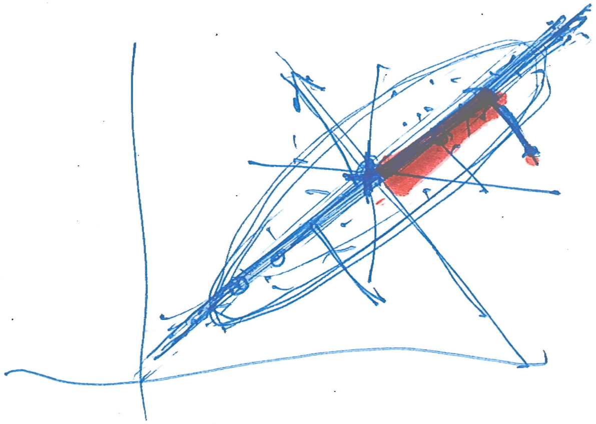


Dimensional

1D subspace



PCA



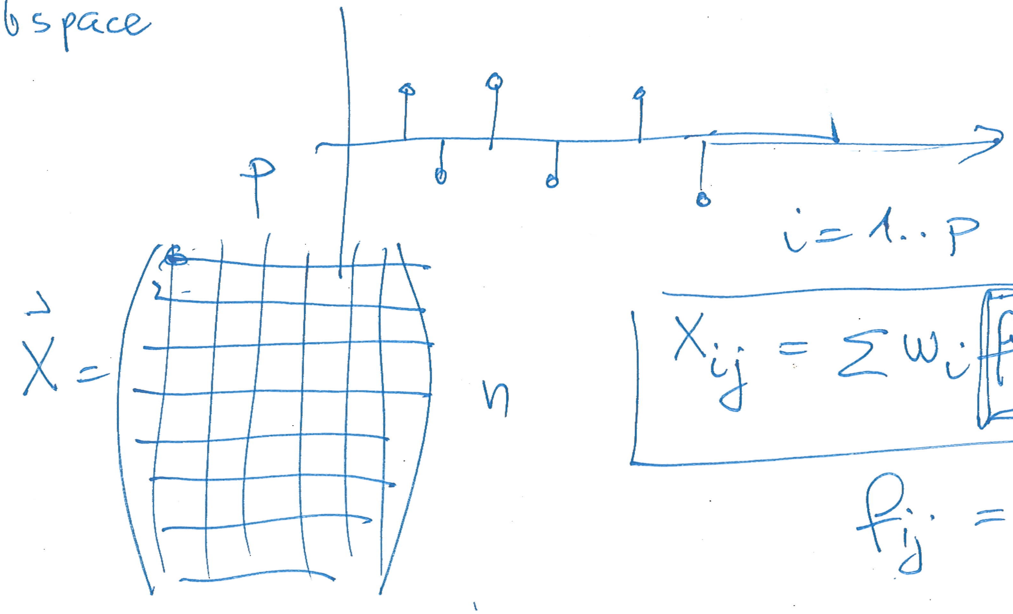
CM coordinates

⇒ subtract the mean.

First PC would be the mean:

$\vec{v}$ subtract project direction

subspace

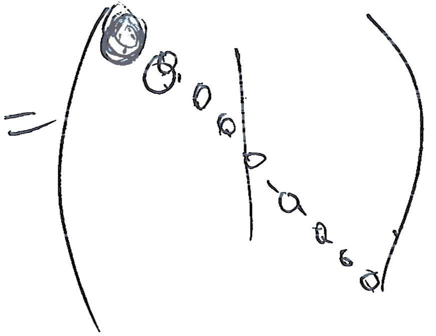


$i = 1..p$ $j = 1..n$

$$\boxed{X_{ij} = \sum w_i \boxed{p_{ij}}}$$

p_{ij} = ith vector at row j

$C_x = X^T X =$



symmetric matrix

$\text{Trace } C_x = \underline{\underline{\sum_i C_{x\,ii}}}$ $\qquad C_x = U \boxed{S} U^T$

$\text{Trace } \underline{\underline{C_x}} = \text{Trace } \underline{\underline{S}} = \sum \lambda_1$

$C_{x11} = (\underbrace{x_1^T \cdot x_1}) = \text{Var}(x_1)$ $\qquad \text{Tr } C_x = \sum_i \text{Var}(x_i)$

<u>Truncation error!</u> $\qquad \sum_{n=K+1}^{N}$ ~~[illegible]~~ λ_n

$$\frac{\sum_{i=1}^{K} \lambda_i}{\sum \lambda_i} = 90-95\%$$