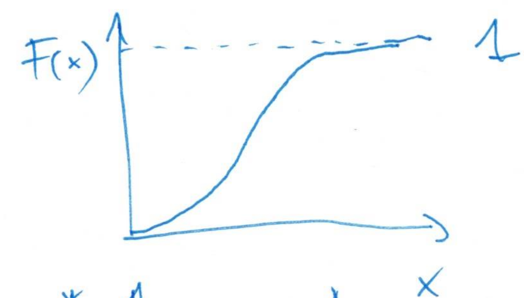
GENERATING RANDOM NUMBERS

(1) Sampling from continuous distributions

INVERSE TRANSFORM METHOD

$x \sim f(x) \Rightarrow$ Cumulative function F.

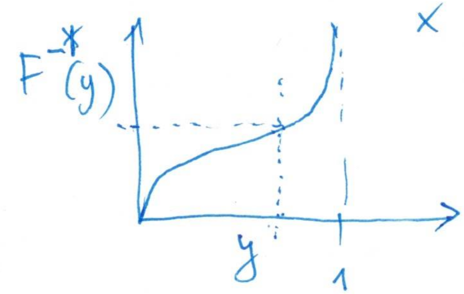
$$F(x<a) = \int_{-\infty}^{a} dx\, f(x)$$



Assume F has an inverse

$$F(x) = y$$

$$x = F^{-1}(y)$$

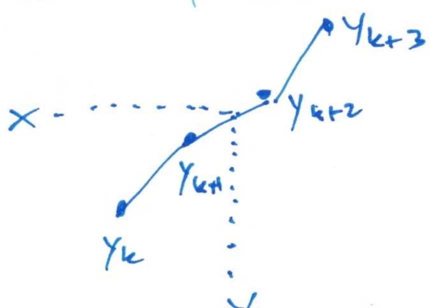


Generate y from $(0,1)$.

Return $x = F^{-1}(y)$

Usual practical approach:

a) Create an array x_i . . . .
b) Calculate a table of $F(x_i) = y_i$
c) Pick a random $y \in [0,1]$
d) Interpolate $x = F(y)$



If there is an analytic inverse (e.g. exponential)

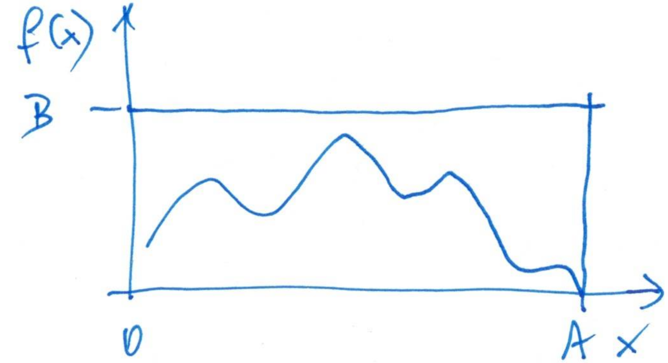
$$F(x) = 1 - e^{-\lambda x} = y$$

$$1 - y = e^{-\lambda x}$$

$$\ln(1-y) = -\lambda x$$

$$\boxed{x = -\frac{\ln(1-y)}{\lambda}}$$

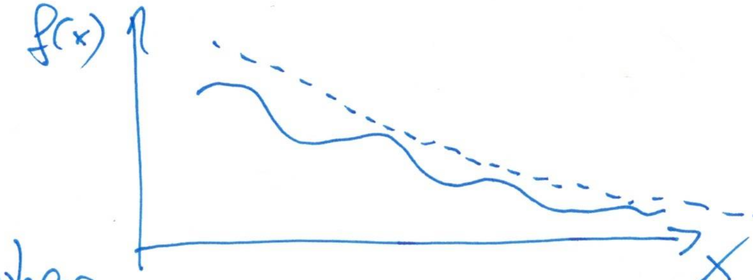
ACCEPTANCE / REJECTION



1) Enclose $f(x)$ in bounding box
2) Generate random $x \in [0, A]$
3) Generate random $y \in [0, B]$
4) Accept y if $y \leq f(x)$

The distribution of x will be $\sim f(x)$.

Sometimes x has infinite support, but $\int_{-\infty}^{\infty} dx\, f(x) < \infty$.



We can then find $g(x) \geq f(x)$ everywhere and has also a finite integral.

$$c = \int dx\, g(x) > 1$$

$$h(x) = \frac{g(x)}{c} \quad \text{density.}$$

1) Generate x with density $h(x)$.
2) Generate $u \in [0, 1]$.
3) Accept if $u \leq f(x)/g(x)$

x will have the correct distribution

CONVOLUTION METHOD

$$y = \sum x_i \, , \qquad \text{each } x_i \sim g_i(x) .$$

Generate x_i separately and return sum.

<u>Example</u>: Erlang distribution (2 params, k, λ)

The sum of independent random exponential variates with the same λ. The distribution of time until the k^{th} event for a Poisson process. (We add the wait times).

$$f(x;k,\lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} .$$

However, it is much easier to generate k exponentials.

GENERATING A NORMAL GAUSS

We can use CLT, but slow.

Box–Muller method:

$$P = \frac{1}{\sqrt{2\pi}} e^{y^2/2} dy \ldots$$

$\boxed{\text{Generating } \underline{\underline{\text{two}}} \text{ Gaussians is easier!}}$

Use the formula from last

$x_1 = r\cos\varphi$

$x_2 = r\sin\varphi$

$$dP(x_1, x_2) = \frac{1}{2\pi} e^{-\frac{x_1^2 + x_2^2}{2}} \cdot dx_1 dx_2$$

$r^2 = x_1^2 + x_2^2$; $\quad dx_1 dx_2 = r\,dr\,d\varphi$..

$$dP = e^{-r^2/2} d\left(\frac{r^2}{2}\right) d\varphi \Rightarrow u = \frac{r^2}{2}$$

$$dP = e^{-u} du . d\varphi$$

Generate u_1 from an exponential distribution.
and φ_1 from $[0, 2\pi]$. $\quad r = \sqrt{2u}$ $\quad x_1 = r\cos\varphi$

$x_2 = r\sin\varphi$

Both will be a normal Gaussian!

DISCRETE DISTRIBUTIONS

Let X be a discrete variable with probabilities

$$P(X = x_i) = p_i \quad ; \quad i = 0, 1 \ldots \quad \sum_{i=0}^{\infty} p_i = 1 .$$

We generate $u \in [0,1]$ and set $X = x_i$, if

$$\sum_{j=0}^{i-1} p_j \le u \le \sum_{j=0}^{i} p_j \Rightarrow$$ Discrete cumulative, we find the containing bin

Requires a search, if we precompute the cumulative array.

$\boxed{\text{Bernoulli}}$ Two outcomes ; 0, 1 $\quad p = P(X=0) = 1 - P(X=1)$

1) Generate $u \in [0,1]$.

$$X = \begin{cases} 0 & \text{if } u < p \\ 1 & \text{else} \end{cases}$$

BINOMIAL

$$p(x=k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

Generate n Bernoulli variates with p

$$x_1, x_2, \ldots x_n$$

$$y = x_1 + \ldots + x_n$$

POISSON

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

X is the number of events in a time interval, if the inter-event times are independent and exponentially distributed with λ.

Generate event times with mean 1. Let j be the smallest index such that

$$\sum_{i=1}^{j+1} x_i > \lambda . \Rightarrow \text{Set } X = j$$

Alternative Poisson:

Generate uniform random variables $u \in [0,1]$

Let j be the smallest index that

$$\prod_{i=1}^{j+1} u_i < e^{-\lambda} \Rightarrow x = j$$